Logarithmic Sobolev Inequalities on Non-isotropic Heisenberg Groups

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Abstract

We study logarithmic Sobolev inequalities with respect to a heat kernel measure on finite-dimensional and infinite-dimensional Heisenberg groups. First we consider logarithmic Sobolev inequalities on non-isotropic Heisenberg groups. These inequalities are considered with respect to the hypoelliptic heat kernel measure, and we show that the logarithmic Sobolev constants can be chosen to be independent of the dimension of the underlying space. In this setting, a natural Laplacian is not an elliptic but a hypoelliptic operator. Furthermore, we apply these results in an infinite-dimensional setting and prove a logarithmic Sobolev inequality on an infinite-dimensional Heisenberg group modelled on an abstract Wiener space.

Non-isotropic Heisenberg Groups $\mathbb{H}_\omega^n$

$\mathbb{H}_\omega^n$ can be regarded as $\mathbb{R}^{2n+1} \cong \mathbb{R}^{2n} \times \mathbb{R}$ equipped with a non-commutative group law given by

$$(\mathbf{v}, z) \cdot (\mathbf{v}', z') := \left(\mathbf{v} + \mathbf{v}', z + z' + \frac{1}{2}\omega(\mathbf{v}, \mathbf{v}')\right),$$
$$(\mathbf{v}, z) = (x_1, \ldots, x_n, y_1, \ldots, y_n, z) \in \mathbb{R}^{2n} \times \mathbb{R},$$
$$(\mathbf{v}', z') = (x_1', \ldots, x_n', y_1', \ldots, y_n' z') \in \mathbb{R}^{2n} \times \mathbb{R}$$
$$\omega : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \longrightarrow \mathbb{R},$$

where

$$\omega(\mathbf{v}, \mathbf{v}') := \sum_{j=1}^{n} \alpha_j (y_j x_j' - x_j y_j')$$

is a symplectic bilinear form on $\mathbb{R}^{2n}$ and

$$0 < \alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_p = \alpha_{p+1} = \cdots = \alpha_n.$$

Sub-Riemannian Structure on $\mathbb{H}_\omega^n$

For any $g \in \mathbb{H}_\omega^n$,

$$X_j^\omega(g) = \frac{\partial}{\partial x_j} + \frac{\alpha_j}{2} y_j \frac{\partial}{\partial z},\ Y_j^\omega = \frac{\partial}{\partial y_j} - \frac{\alpha_j}{2} x_j \frac{\partial}{\partial z},\ j = 1, \cdots, n.$$

Hörmander's condition:

$$[X_j^\omega, Y_j^\omega] = -\alpha_j \frac{\partial}{\partial z},\ j = 1, \cdots, n.$$

$\mathbb{H}_\omega^n$ admits a sub-Riemannian structure $(\mathbb{H}_\omega^n, \mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}}^\omega)$ where $\mathcal{H}_g = Span\{X_j^\omega(g), Y_j^\omega(g) : j = 1, \cdots, n\}$ for any $g \in \mathbb{H}_\omega^n$ and $\{X_j^\omega, Y_j^\omega : j = 1, \cdots, n\}$ is an orthonormal frame for $\mathcal{H}$.

Sub-Laplacian $\Delta_{\mathcal{H}}^\omega$ and Heat Kernel Measure $\{\mu_t^\omega\}_{t>0}$

Horizontal gradient:

$$\nabla_{\mathcal{H}}^\omega f := \sum_{i=1}^{n} \left((X_i^\omega f) X_i^\omega + (Y_i^\omega f) Y_i^\omega\right),\ f \in C^\infty(\mathbb{H}_\omega^n).$$

Sub-Laplacian:

$$\Delta_{\mathcal{H}}^\omega := \sum_{j=1}^{n} \left[(X_j^\omega)^2 + (Y_j^\omega)^2\right].$$

$\Delta_{\mathcal{H}}^\omega$ is hypoelliptic.

(Hypoelliptic) Heat kernel measure $\{\mu_t^\omega\}_{t>0}$:
a family of probability measure with

$$d\mu_t^\omega = p_t^\omega(g) dg$$

where $p_t^\omega(g)$ is the hypoelliptic heat kenrel for $\Delta_{\mathcal{H}}^\omega$ and $dg = dx_1 dy_1 \cdots dx_n dy_n dz$ is (the) Haar measure on $\mathbb{H}_\omega^n$.

Result 1 (on $\mathbb{H}_\omega^n$)

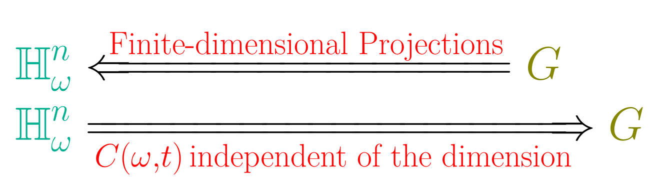
Theorem:(Theorem 4.5 in [1])

- For $f \in C_c^\infty(\mathbb{H}_\omega^n)$ and $t > 0$,

$$\int_{\mathbb{H}_\omega^n} f^2 \log f^2 d\mu_t^\omega - \left(\int_{\mathbb{H}_\omega^n} f^2 d\mu_t^\omega\right) \log\left(\int_{\mathbb{H}_\omega^n} f^2 d\mu_t^\omega\right) \leqslant C(\omega, t) \int_{\mathbb{H}_\omega^n} |\nabla_{\mathcal{H}}^\omega f|^2 d\mu_t^\omega.$$

- $C(\omega, t) = C(\omega_0) t$ and $C(\omega_0)$ is the logarithmic Sobolev constant on the isotropic Heisenberg group $\mathbb{H}_{\omega_0}^1$.
- $C(\omega, t)$ is independent of the dimension of $\mathbb{H}_\omega^n$!

Finite-dimensional Projection Approximation



Result 2 (on G)

Theorem: (Theorem 6.24 in [1])

- For $f \in \mathcal{D}(\mathcal{E}_t)$ and $t > 0$

$$\int_G f^2 \log f^2 d\nu_t - \left(\int_G f^2 d\nu_t\right) \log\left(\int_G f^2 d\nu_t\right) \leqslant C(\omega, t) \mathcal{E}_t(f, f).$$

- $C(\omega, t)$ is independent of ω!

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Introduction

In [4], Gross introduced and studied the following dimension-free logarithmic Sobolev inequality, i.e.

$$\int_{\mathbb{R}^n} f^2 \log f^2 d\mu - \left(\int_{\mathbb{R}^n} f^2 d\mu\right) \log\left(\int_{\mathbb{R}^n} f^2 d\mu\right) \leqslant 2 \int_{\mathbb{R}^n} |\nabla f|^2 d\mu$$

for any $f \in C_c^\infty(\mathbb{R}^n)$ where $d\mu(x) = \frac{e^{-\frac{|x|^2}{2}}}{(2\pi)^{\frac{n}{2}}} dx$ is the Gaussian measure on $\mathbb{R}^n$.
In the hypoelliptic setting, a logarithmic Sobolev ineuqality was first proved in [6] on the three-dimensional standard Heisenberg group. Later it was generalized to higher-dimensional settings in [3], [5], etc.
However, the (in)dependence of the logarithmic Sobolev constant on the dimension is not known in neither the standard nor the non-isotropic setting.

Infinite-dimensional Heisenberg-like Groups G

Let (W, H, μ) be a real abstract Wiener space.
The *infinite-dimensional Heisenberg-like group* with a one-dimensional center G can be regarded as $W \times \mathbb{R}$ with a non-commutative group law given by

$$(w_1, c_1) \cdot (w_2, c_2) = \left(w_1 + w_2, c_1 + c_2 + \frac{1}{2}\omega(w_1, w_2)\right),$$
$$(w_i, c_i) \in W \times \mathbb{R},\ i = 1, 2,$$
$$\omega : W \times W \to \mathbb{R}$$

where ω is a continuous skew-symmetric bilinear form on W.

Subelliptic Laplacian L and Heat Kernel Measure ν_t

Let $\{e_j\}_{j=1}^\infty$ be an orthonormal basis for H.
Subelliptic Laplacian:

$$Lf(x) := \sum_{j=1}^{\infty} \left[\widetilde{(e_j, 0)}^2 f\right](x)$$

for any *cylinder function* $f : G \to \mathbb{R}$.
Horizontal gradient:
$\operatorname{grad}_H u : G \to H$ of any *cylinder polynomials* u is defined by

$$\langle \operatorname{grad}_H u, h \rangle_H = \widetilde{(h, 0)} u,\ h \in H.$$

Heat Kernel Measure:

$$\nu_t = \operatorname{Law}(g_t)$$

for any $t > 0$ and g_t is the Brownian motion generated by $\frac{1}{2}L$ on G.
Dirichlet Form $\mathcal{E}_t$:
the closure of

$$\mathcal{E}_t^0(u, v) := \int_G \langle \operatorname{grad}_H u, \operatorname{grad}_H v \rangle_H d\nu_t.$$

on $L^2(G, \nu_t)$ where u, v are cylinder polynomials.

Future Directions

Step-2 Carnot groups

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